Option Pricing and Arbitrage

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1 Overview

We shall show that there is a wide class of financial instruments and securities whose price does not depend on an economic agent's assessment of probabilities of future events or on his risk-aversion, but instead can be determined uniquely from the prices of basic securities already marketed. Essentially this is possible, because the stream of payments into and outflows from the security, necessitated by holding it, can be manufactured by trading in the marketed securities, starting with a particular portfolio. The value of the security is then simply the current market value of that initial portfolio: any less and an agent could buy the security and sell the initial portfolio short (have a negative holding), which means that all future net capital flows are zero, and an instant profit of the price difference has been made. Were the security price greater than the portfolio's price, one would sell the security short and buy the portfolio for a similar result. The key economic assumption throughout shall be that there is an absence of such arbitrage opportunities — that you cannot get 'something for nothing'.

Building up from the discrete case, we shall firstly consider, as Black-Scholes did in their atavistic paper [1], a market of a riskless deterministic bond and a single stock, whose log-price is a Brownian diffusion, and the security being a European call option. Great generalisations are possible, due to Harrison and Pliska [3] using Itô calculus, enabling valuation of different derivative securities, and with different price processes. Duffie [2] points the way to further economic generalisations and other traded futures contracts. Throughout we shall use economic assumptions relating to perfect markets, such as no transaction costs or taxation, and the unlimited ability to sell short (and to buy). Also, we always work with a finite horizon — we shall always stop at time T. This makes some of our stochastic calculus easier, but is a fairly realistic assumption.

Simple mathematics in the discrete case will build up to a rich theory in the continuous case, where stochastic calculus will be relied on heavily for the manipulation, and even definition, of the processes involved — price, gain, and portfolio management processes. Finally, we will consider the implication of the financial and mathematical assumptions made during this discourse.

2 Specific Discrete Case

Most of what follows is adapted from Duffie [2, Chap 22], and is of a motivational and explanatory nature.

Consider the Land of Nod, in which every year is either *good* or *bad*, according to an (unknown) probabilistic law. Suppose that there are two marketed securities: the *drachma*, whose value increases by 5% each year regardless, and the *obol* whose value increases by 10% in a good year, and remains constant in a bad one. Suppose now that a new security, the *lepta*, is introduced, which pays l_g in a good year and l_b in a bad one. Pick α, β in $\mathbb{R}$ such that

$$\begin{aligned} 1.10\alpha + 1.05\beta &= l_g \\ 1.00\alpha + 1.05\beta &= l_b \end{aligned} \tag{2.1}$$

That is

$$\alpha = 10(l_g - l_b) \qquad \beta = \frac{20}{21}(11 l_b - 10 l_g)$$

$$\text{So} \qquad \alpha + \beta = \frac{1}{1.05}\left(\tfrac{1}{2} l_g + \tfrac{1}{2} l_b\right) \tag{2.2}$$

Then we see that by buying α *obols* and β *drachmae*, in a good year we get l_g and in a bad one l_b. That is, we have manufactured the *lepta* by trading in the other two securities. This crucial idea of manufacturing securities from other securities is due to Black-Scholes [1]. We thus see that the only possible price for a *lepta* is $\alpha+\beta$ (assuming the others have unit price). For if it were less, we would buy a *lepta*, and borrow (sell short) α *obols* and β *drachmae*. Whether the next year were good or bad, the value of the *lepta* would enable us to pay back α *obols* and β *drachmae* at their new value, and keep an arbitrage profit. Note that (2.2) looks like probabilities of $\frac{1}{2}$ each have been assigned to good and bad years and $\alpha+\beta$ is the discounted (at *drachma* rate) expectation of the value of a *lepta*. In other words the (discounted) value of a *lepta* is a martingale under the probability distribution $(\frac{1}{2}, \frac{1}{2})$. This result will generalise and become central both in the discrete and continuous cases.

For now, note that this distribution depends not on the actual probabilities, but on the securities. For if, say, the *drachma* goes up by a factor of r every year, and the *obol* goes up by a factor of p_g in a good year, and by p_b in a bad year ($p_g \neq p_b$), then

$$\alpha+\beta = \frac{1}{r}\left(\frac{r-p_b}{p_g-p_b} l_g + \frac{p_g-r}{p_g-p_b} l_b\right) \tag{2.3}$$

Which, for $p_b < r < p_g$, is a convex combination of l_g and l_b, that is the expectation under artifical probability distribution $((r-p_b)/(p_g-p_b), (p_g-r)/(p_g-p_b))$, discounted by the riskless rate. (The ordering of p_b, r, p_g is due to economic assumptions — any other order would result in one of the two securities being always inferior and never traded.)

Suppose now that the *lepta* is a security whose value is determined after two years, and is l_{gg} after two good years, l_{gb} after a good year followed by a bad year, and l_{bg}, l_{bb} in the other cases. Setting

$$q_g = \frac{r-p_b}{p_g-p_b} \qquad q_b = \frac{p_g-r}{p_g-p_b} \tag{2.4}$$

to be the artifical probabilities derived in (2.3), we see that the value of a *lepta* after one year is:

$$\text{value} \;=\begin{cases} \frac{1}{r}(q_g l_{gg} + q_b l_{gb}) & \text{if first year good} \\ \frac{1}{r}(q_g l_{bg} + q_b l_{bb}) & \text{if first year bad.}\end{cases} \tag{2.5}$$

So deduce the value of a *lepta* at the start is:

$$\frac{1}{r^2}(q_g^2 l_{gg} + q_g q_b l_{gb} + q_b q_g l_{bg} + q_b^2 l_{bb}) \tag{2.6}$$

$$\text{or} \qquad \frac{1}{r^2}\mathbb{E}_Q(L)$$

where L is the (random) worth of the *lepta* after two years, and Q is the artificial probability law, under which each year is IID: good with probability q_g, bad with probability q_b.

We see that this generalises, so that any security X, which is $\mathcal{F}_n$-measurable ($\mathcal{F}_n$ is the filtration induced by the results of the years) has value now of:

$$r^{-n}\mathbb{E}_Q(X) \tag{2.7}$$

By value here, we mean as usual, that if someone were prepared to trade in security X for either more or less than this price, an arbitrage profit could be made by manufacturing X by trading in a portfolio of *drachmae* and *obols*, and selling either X or the portfolio short, against a holding in the other.

Note also that V_t, the value of the security at time t is:

$$\begin{aligned} V_t &= r^{-(n-t)}\mathbb{E}_Q(X \mid \mathcal{F}_t) \\ r^{-t}V_t &= \mathbb{E}_Q(r^{-n}X \mid \mathcal{F}_t) \end{aligned} \tag{2.8}$$

That is, $r^{-t}V_t$ is a martingale under the law Q. We shall meet this idea again.

3 General Discrete Case

We consider the general case, as set out in Harrison and Pliska [3, §2], with probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The sample space Ω is finite, all $\omega \in \Omega$ have positive probabilities, we stop at time $T \in \mathbb{N}$, and we have a filtration $(\{\mathcal{F}_t\} : t = 0, \ldots, T)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{P}(\Omega)$.

We posit a price process $S = (S_t)_{t=0}^T$, $S_t = (S_t^0, S_t^1, \ldots, S_t^K) \in \mathbb{R}^{K+1}$, such that S_t is $\mathcal{F}_t$-measurable, and $S_t^i > 0$, and that $\mathcal{F}_t$ is the information known to everyone at time t, that is only the past and present prices are known. We call the (S^0) process the *bond*, but it has no special properties from the others, though it will have in continuous time (except without loss, we assume that $S_0^0 = 1$).

We assume that one, an economic agent, has a *trading strategy* ϕ, which is a previsible stochastic process of portfolios:

$$\phi_t = (\phi_t^0, \ldots, \phi_t^K) \qquad \phi_t \text{ is } \mathcal{F}_{t-1}\text{-measurable} \qquad 1 \le t \le T \tag{3.1}$$

Here, ϕ_t^i represents the agent's holding of security i between times $t-1$ and t, depending of course only on prices up to time $t-1$. We allow each ϕ^i to be unbounded in both directions, that is allowing unlimited short-selling (negative holdings) and also implies an infinite share capital. Later we shall consider bounds on ϕ given the security to be valued.

At time t, the agent will change his portfolio from ϕ_t to ϕ_{t+1}, using $\mathcal{F}_t$ information only. We say the strategy is *self-financing* if there is no net cost in doing so, that is if:

$$\phi_t . S_t = \phi_{t+1} . S_t \qquad (1 \le t \le T-1) \tag{3.2}$$

Implicit in this statement is a complete absence of transaction costs, an assumption we shall lean on heavily.

The *value process* $V(\phi)$ is defined:

$$V_t(\phi) = \begin{cases} \phi_t . S_t & 1 \le t \le T \\ \phi_1 . S_0 & t = 0 \end{cases} \tag{3.3}$$

and describes the value of the portfolio just after the $\mathcal{F}_t$ information is released at time t. So we can rewrite condition (3.2) as

$$V_t - V_{t-1} = \Delta V_t = \phi_t . \Delta S_t \qquad (1 \le t \le T) \tag{3.4}$$

It is common to restrict attention to a set of trading strategies described as *admissible*, usually meaning self-financing and such that $V_t(\phi) \ge 0$ almost surely for $0 \le t \le T$, though Harrison & Pliska [3] show that in the absence of arbitrage this can be weakened to $V_T(\phi) \ge 0$.

To justify making assumptions about the existence of martingale measures in the continuous case, they also point out a relationship between the following definitions: a *(contingent) claim* is an $\mathcal{F}_T$-measurable non-negative random variable X, which is *attainable* (in Duffie *redundant*) if there exists an admissible trading strategy ϕ such that $V_T(\phi) = X$. An *arbitrage opportunity* is an admissible trading strategy ϕ such that $V_0(\phi) = 0$, but $\mathbb{E}(V_T(\phi)) > 0$. Set $\mathcal{P}$ to be the set of probability measures Q equivalent to $\mathbb{P}$ such that $(\beta_t S_t)$ is a vector martingale under Q, where $\beta_t = 1/S_t^0$, the *discount process*. They then show:

Theorem. *There are no arbitrage opportunities if and only if $\mathcal{P}$ is non-empty, in which case there is a unique price for an attainable claim X given by:*

$$\mathbb{E}_Q(\beta_T X) \qquad \text{for any and every } Q \in \mathcal{P} \tag{3.5}$$

We recognise this as a generalisation of what was observed in Section 2, where there $S_t^0 = r^t$ and $\beta_t = r^{-t}$. It is instructive to see the proof of this result. (Though we shall simplify a little by taking the weaker definition of admissibility.)

Call π a *price system* if $\pi : \mathrm{m}\mathcal{F}_T^+ \longrightarrow \mathbb{R}^+$ such that:
(1) $\pi(X) = 0 \iff X = 0$
(2) π is semilinear, that is $\pi(\alpha X + \beta Y) = \alpha\pi(X) + \beta\pi(Y)$ for α, β positive.
(3) for an admissible trading strategy ϕ, $\pi(V_T(\phi)) = V_0(\phi)$.

Lemma. *There is a bijection between $\mathcal{P}$ and Π, the set of all price systems, given by:*

$$\begin{aligned} \mathcal{P} &\longleftrightarrow \Pi \\ Q &\longmapsto X \mapsto \mathbb{E}_Q(\beta_T X) \\ A \mapsto \pi(S_T^0 I_A) &\longleftarrow \pi \end{aligned} \tag{3.6}$$

Proof of lemma

Given $Q \in \mathcal{P}$ set $\pi(X) = \mathbb{E}_Q(\beta_T X)$, and then clearly π satisfies (1) and (2). Given an admissible ϕ,

$$\pi(V_T(\phi)) = \mathbb{E}_Q(\phi_T.\beta_T S_T) = \mathbb{E}_Q(\phi_T.\mathbb{E}_Q(\beta_T S_T \mid \mathcal{F}_{T-1})) = \mathbb{E}_Q(\phi_T.\beta_{T-1} S_{T-1})$$

as βS is a Q-martingale. The self-financing relation (3.2), and induction show

$$\ldots = \mathbb{E}_Q(\phi_{T-1}.\beta_{T-1} S_{T-1}) = \mathbb{E}_Q(\phi_0.\beta_0 S_0) = V_0(\phi)$$

by the triviality of $\mathcal{F}_0$, so π is a price system.

Conversely, given $\pi \in \Pi$, set $Q(A) = \pi(S_T^0 I_A)$. Then (1) implies that Q is equivalent to $\mathbb{P}$, and (2) implies that Q is a measure. Using (3) with $\phi_t^0 = 1$ and other ϕ^i zero, then $V_T(\phi) = S_T^0$, and we see that $Q(\Omega) = 1$. Given $X \in \mathrm{m}\mathcal{F}_T^+$,

$$\mathbb{E}_Q(\beta_T X) = \sum_{\omega\in\Omega} Q(\omega)\beta_T(\omega)X(\omega) = \sum_{\omega\in\Omega} \pi(S_T^0 I_\omega)\beta_T(\omega)X(\omega)$$

$$= \pi\left(\sum_{\omega\in\Omega} S_T^0 I_\omega X(\omega)\beta_T(\omega)\right) = \pi(S_T^0\beta_T X) = \pi(X) \tag{3.7}$$

To show that βS is a Q-martingale, suppose τ is a stopping time, fix k, and set ϕ equal to zero except:

$$\phi_t^k = \begin{cases} 1 & t \le \tau \\ 0 & t > \tau \end{cases} \qquad \phi_t^0 = \begin{cases} 0 & t \le \tau \\ S_\tau^k / S_\tau^0 & t > \tau \end{cases} \tag{3.8}$$

Check that ϕ is an admissible strategy, then $V_T(\phi) = S_T^0 \beta_\tau S_\tau^k$, and $V_0(\phi) = S_0^k = \beta_0 S_0^k$. Then using (3.7) and condition (3),

$$\mathbb{E}_Q(\beta_\tau S_\tau^k) = \mathbb{E}_Q(\beta_T V_T(\phi)) = \pi(V_T(\phi)) = V_0(\phi) = \beta_0 S_0^k$$

By Rogers [6, 1.2], βS is a martingale under Q. It is easy to see, using (3.7), that the given maps are mutual inverses. □

Proof of Theorem

Now we see that if there is a price system π and given ϕ admissible with $\mathbb{E}V_T(\phi) > 0$, then $V_T(\phi) \neq 0$, and so $\pi(V_T(\phi)) = V_0(\phi) \neq 0$, so ϕ is not an arbitrage opportunity.

Conversely, if there are no arbitrage opportunities, we set

$$\mathcal{X}^+ = \{\, X \in \mathrm{m}\mathcal{F}_T^+ : \mathbb{E}(X) \ge 1 \,\} \quad \text{a closed convex cone in } \mathbb{R}^\Omega$$
$$\mathcal{X}^0 = \{\, X \in \mathrm{m}\mathcal{F}_T : X = V_T(\phi)\ \phi \text{ self-financing}, V_0(\phi) = 0 \,\}$$

Note that ϕ above need not be admissible, and that $\mathcal{X}^0$ is a linear subspace of $\mathbb{R}^\Omega$. Absence of arbitrage opportunities implies that $\mathcal{X}^+ \cap \mathcal{X}^0 = \emptyset$, so the strict separating hyperplane theorem gives us a linear functional P on $\mathbb{R}^\Omega$, $P = 0$ on $\mathcal{X}^0$ and $P \ge 1$ on $\mathcal{X}^+$. Define π on $\mathrm{m}\mathcal{F}_T^+$ by $\pi(X) = P(X)/P(S_T^0)$. Replacing X by $X/\mathbb{E}X$ if necessary, we see that $P(X) \ge \mathbb{E}(X)$ for non-negative X, so that π is well-defined and $\pi(S_T^0) = 1$. Clearly π satisfies conditions (1) and (2) of the lemma. Given an admissible trading strategy ϕ, define ψ by $\psi_t^i = \phi_t^i - \delta_{i0} V_0(\phi)$, then $V_0(\psi) = 0$ and ψ is self-financing. So $V_T(\psi) \in \mathcal{X}^0$, thus $\pi(V_T(\psi)) = 0$. But

$$\pi(V_T(\psi)) = \pi(V_T(\phi)) - \pi(V_0(\phi)S_T^0) = \pi(V_T(\phi)) - V_0(\phi) = 0 \tag{3.9}$$

Given an attainable contingent claim X, with admissible ϕ such that $V_T(\phi) = X$, then the value of the claim at time 0 must be

$$V_0(\phi) = \pi(V_T(\phi)) = \mathbb{E}_Q(\beta_T X) \tag{3.10}$$

where Q is any measure under which (βS) is a martingale. If X were offered for sale for less, one could short sell (borrow) the portfolio and use some, but not all, of the proceeds to buy the claim X now. Then by time T, the negative holding incurs a liability of X, which is exactly matched by the realised claim. This leaves a risk-free profit of the difference in prices (at time 0 prices). Conversely, if agents are willing to buy claim X for more than the price (3.10), we would use that money now to buy a portfolio (with some left over) which would supply the agent at time T with exactly our liability X to him.

The absence of arbitrage, in both cases, says that this cannot happen. $\square$

The question of claim pricing in the discrete case now becomes one of characterising the claims which are attainable. Discrete results have little bearing on the more important continuous case, so we shall ignore them. Remarking only that Harrison & Pliska [3] discuss results based on the fineness of $\mathcal{F}_t$ at each stage, and that in the Land of Nod scenario (section 2), the *obols* and *drachmae* are sufficient to 'span' the entire set of claims maturing at any given finite time.

4 Continuous Case — Black-Scholes

Historically the first case to be studied was not the discrete case (simple though that is), but a specification of the continuous case. In their paper of 1973, Black and Scholes [1] considered a two-security market of stock and bond. Their further assumptions on the price processes were:

$$\begin{aligned} \text{bond}: \quad S_t^0 &= e^{rt} \\ \text{stock}: \quad S_t^1 &= e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t} \end{aligned}$$

where B is a Brownian motion relative to $(\Omega, \mathcal{F}, \mathbb{P})$. That is to say, the bond has a constant deterministic rate of return, and the stock has a rate of return which is drifting Brownian motion:

$$dS_t^1 = S_t^1(\sigma\, dB_t + \mu t) \qquad \text{(by Itô)} \tag{4.1}$$

Black-Scholes then derived their famous differential equation, albeit by imprecise methods:

$$\frac{\partial C}{\partial t}(x,t) = rC(x,t) - rx\frac{\partial C}{\partial x}(x,t) - \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t) \tag{4.2}$$

where $C(S_t^1, t)$ is the value of a claim at time t, with boundary condition $C(S_T^1, T) = X$, where X is the claim ($\mathcal{F}_T$ random variable) to be valued. This equation is correct, but we shall derive it by other means. Their claim was $(S_T^1 - c)^+$, a *European call option*, exercisable at time T at price c.

The method presented in this section captures more of the spirit of the original work, whilst the method of the next section is more closely connected with the generalised continuous case. Both are again based on Black-Scholes' idea of manufacturing the claim by trading in the two basic securities, and letting the no-arbitrage assumption force the market price to be the initial portfolio investment required to manufacture the claim. The difference lies in how this strategy is computed.

Bootstraps

If we posit the existence of a $C^{2,1}_{\mathbb{R}^+}(\mathbb{R}^+, [0,T])$ function C, which values the claim at time t as $C(S_t^1, t)$, then we define a strategy ϕ using C, and see that for ϕ to be self-financing, (4.2) is sufficient. Solving the PDE we see that such a solution exists. So we have attained the claim using an admissible trading strategy, so its value at any instant (by the arbitrage argument) is simply the value of the portfolio at that moment, which we shall see really is $C(S_t^1, t)$.

So, suppose $C(\cdot,\cdot) \in C^{2,1}_{\mathbb{R}^+}(\mathbb{R}^+, [0,T])$ is the value function of the claim. Then define

$$\begin{aligned} \phi_t^1 &= \frac{\partial C}{\partial x}(S_t^1, t) \\ \phi_t^0 &= \left(C(S_t^1, t) - S_t^1 \frac{\partial C}{\partial x}(S_t^1, t)\right)/S_t^0 \end{aligned} \tag{4.3}$$

So that $\phi_t.S_t := \phi_t^0 S_t^0 + \phi_t^1 S_t^1 = C(S_t^1, t)$. We see that ϕ is continuous and adapted. We say ϕ is *self-financing* by analogue with the discrete case (3.4) if $d(\phi.S)_t = \phi_t.dS_t$. Which means, we need

$$d(C(S_t^1, t)) = \phi.dS_t$$

$$dC_t = \frac{\partial C}{\partial x}\, dS_t^1 + \frac{\partial C}{\partial t}\, dt + \tfrac{1}{2}\frac{\partial^2 C}{\partial x^2}\, d[S^1]_t \qquad \text{(Itô)}$$

$$\text{now} \qquad dS_t^1 = S_t^1(\sigma\, dB_t + \mu t), \qquad d[S^1]_t = \sigma^2 (S_t^1)^2\, dt$$

$$\text{Thus}\quad dC_t = \frac{\partial C}{\partial x} S_t^1 \sigma\, dB_t + \left(\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 (S_t^1)^2 \frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial x} S_t^1 \mu \right) dt$$

$$\phi_t . dS_t = \left(C - S_t^1 \frac{\partial C}{\partial x} \right) / S_t^0 . (r S_t^0\, dt) + \frac{\partial C}{\partial x} S_t^1 (\sigma\, dB_t + \mu t)$$

as $dS_t^0 = r S_t^0\, dt$. Cancelling terms, we see that $d(\phi . S) = \phi . dS$ if and only if

$$\frac{\partial C}{\partial t}(S_t^1, t) + \tfrac{1}{2}\sigma^2 {S_t^1}^2 \frac{\partial^2 C}{\partial x^2} = r C(S_t^1, t) - r S_t^1 \frac{\partial C}{\partial x} \tag{4.4}$$

This will be true if the Black-Scholes PDE (4.2) holds. That PDE can be solved (by change of variable, and then Fourier transform) to derive the *Black-Scholes formula*:

$$C(x,t) = x \Phi \left(\frac{\log \frac{x}{c} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) - c e^{-r(T-t)} \Phi \left(\frac{\log \frac{x}{c} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \tag{4.5}$$

where $\Phi(\alpha) = \mathbb{P}(\, \mathrm{N}(0,1) \leq \alpha)$, is the normal distribution function. We can see that this satisfies $C(x,T) = (x-c)^+$, so that $C(S_T^1, T) = (S_T^1 - c)^+ = X$. The existence of a smooth solution C implies that the assumptions we made were valid and a self-financing trading strategy does exist.

Points to note

(1) Writing $C_t(x)$ to be $C(x, T-t)$ for $t \geq 0$, the value of the claim with time t till maturity. We shall show shortly that $\frac{\partial}{\partial t} C_t(x) > 0$, that is we always hope to do better in the longer term than the shorter term. As $C_0(x) = (x-c)^+$ and $C_\infty(x) = x$, we deduce that $(x-c)^+ \leq C_t(x) \leq x$. In fact, the picture is of a family of convex curves.

(2) We see also that $(x - C_t(x)) \to c e^{-rt}$, as $x \to \infty$. A calculation also shows that

$$\frac{\partial}{\partial x} C_t(x) = \Phi \left(\frac{\log \frac{x}{c} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right) \tag{4.6}$$

which is increasing in x and lies in $[0,1]$. Thus we deduce that C_t is convex in x, and more importantly that $\phi_t^1 \in [0,1]$ for all t. That is, we always hold a non-negative and bounded amount of the stock. The value of the bond holding or borrowing is never more than c away from the stock price S^1.

(3) If we write

$$\alpha = \frac{\log \frac{x}{c} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \qquad \beta = \frac{\log \frac{x}{c} + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

then $C = x\Phi(\alpha) - c e^{-rt}\Phi(\beta)$, $\frac{\partial C}{\partial x} = \Phi(\alpha)$, and

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{\sigma x \sqrt{2\pi t}} e^{-\frac{1}{2}\alpha^2}$$

Thus using (4.2),

$$\begin{aligned} \frac{\partial C}{\partial t} &= -r x \Phi(\alpha) + r c e^{-rt}\Phi(\beta) + r x \Phi(\alpha) + \frac{\sigma x}{2\sqrt{2\pi t}} e^{-\frac{1}{2}\alpha^2} \\ &= \frac{\sigma x}{2\sqrt{2\pi t}} e^{-\frac{1}{2}\alpha^2} + r c e^{-rt}\Phi(\beta) > 0 \end{aligned}$$

As claimed earlier.

(4) $C_t(x)$ is independent of μ, the initial Brownian drift, or underlying rate of return of the stock. We shall see why in the next section, as we change measure.

5 Black-Scholes Generalised

We now consider an alternative approach, which gives the same answer as section 4 for the basic Black-Scholes model, but also generalises the distributional assumption. The distribution is now stated, and then discussed briefly:

B is CBM($\mathbb{R}^K$) on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is Weiner measure, and $\mathcal{F}$ has usual filtration $(\mathcal{F}_t : 0 \leq t \leq T)$. The price process $\bar{S} = (S^0, S)$, where $S = (S^1, \ldots, S^K)$ and

$$S_t^0 = \exp\left(\int_0^t \rho(s)\, ds\right)$$

for a deterministic continuous function $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}$. And S is defined by the stochastic differential equation:

$$dS_t = \mu_0(S_t, t)\, dt + \sigma_0(S_t, t)\, dB_t \tag{5.1}$$

where $\mu_0 : \mathbb{R}^K \times \mathbb{R}^+ \longrightarrow \mathbb{R}^K$, $\sigma_0 : \mathbb{R}^K \times \mathbb{R}^+ \longrightarrow \mathcal{L}(\mathbb{R}^K, \mathbb{R}^K)$ are Lipschitz continuous in x (the first coordinate). We further assume that

$$\begin{aligned} \mu_0(x,t) &= \operatorname{diag}(x^1, \ldots, x^K)\mu_1(x,t) \\ \sigma_0(x,t) &= \operatorname{diag}(x^1, \ldots, x^K)\sigma_1(x,t) \end{aligned}$$

where σ_1 is positive definite, and μ_1, σ_1 and σ_1^{-1} are bounded in x and t ($x \gg 0$), and continuous by necessity.

Notes on assumptions

(1) μ_0, σ_0 are Lipschitz and satisfy a growth condition, so standard SDE theory [7, V.13.1] gives a unique strong solution to (5.1), which is continuous in t, for any constant vector S_0.

(2) We need the boundedness of μ_1 and σ_1^{-1} to achieve an explicit martingale for a change of measure.

(3) Black-Scholes is recovered for $K = 1$, $\mu_1(x,t) = \mu$, $\sigma_1(x,t) = \sigma$ and $\rho(t) = r$.

(4) For $\mu_1(x,t) = \mu$, $\sigma_1(x,t) = \sigma$, σ a positive-definite $K \times K$ matrix, then

$$S_t^i = S_0^i \exp\{\sigma_{ij} B_t^j + (\mu_i - \frac{1}{2}\sum_j \sigma_{ij}^2)t\} \qquad \text{as in [3, §5]}$$

(5) This is scarcely more general than Rogers [6], the only difference being the dependence of μ_1 and σ_1 on x. It is difficult to extend further— Duffie [2, §22K] claims to do so using previous results. However those results give rise to a new measure $\mathbb{Q}$ such that $d\mathbb{Q}/d\mathbb{P}$ is bounded, which is not what occurs, even with our tight constraints. His basic approach, though, is unflawed and it inspires what follows.

Define $\beta_t = 1/S_t^0$, the *discount process*, which is a finite variation continuous process. Set $Z_t = \beta_t S_t$, a continuous $\mathbb{R}^K$-valued semimartingale. Using integration by parts:

$$\begin{aligned} dZ_t &= d\beta_t\, S_t + \beta_t\, dS_t + d[\beta, S]_t \\ &= (-\rho(t)\beta_t\, dt)S_t + \beta_t(\mu_0\, dt + \sigma_0\, dB_t) + 0 \\ &= (\beta_t\mu_0 - \rho(t)Z_t)dt + \beta_t\sigma_0\, dB_t \end{aligned} \tag{5.2}$$

Set

$$\begin{aligned} \mu(x,t) &:= \mu_1(x/\beta_t, t) - \rho(t)1_K \qquad 1_K = (1, \ldots, 1) \in \mathbb{R}^K \\ \sigma(x,t) &:= \sigma_1(x/\beta_t, t) \end{aligned} \tag{5.3}$$

Then

$$dZ_t = \operatorname{diag}(Z_t^1, \ldots, Z_t^K)\,\{\mu(Z_t, t)\, dt + \sigma(Z_t, t)\, dB_t\} \tag{5.4}$$

Note that ρ being bounded implies that μ and σ^{-1} are bounded. Set $\theta_t := -\sigma^{-1}(Z_t, t)\mu(Z_t, t)$, which is a continuous adapted bounded $\mathbb{R}^K$-valued process. Then the Cameron-Martin-Girsanov theorem [7, IV.38.9] says that

$$\xi_t = \mathcal{E}(\theta \cdot B)_t = \exp\left\{\int_0^t \theta_s . dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 \, ds\right\} \tag{5.5}$$

is a well-defined continuous martingale and there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ with

$$\begin{aligned} \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} &= \xi_t \\ \text{and} \qquad \tilde{B}_t &= B_t - \int_0^t \theta_s \, ds \end{aligned} \tag{5.6}$$

is a Brownian motion under $\mathbb{Q}$. Then

$$dZ_t = \operatorname{diag}(Z_t^1, \ldots, Z_t^K)\sigma(\sigma^{-1}\mu \, dt + dB_t) = \operatorname{diag}(Z_t^1, \ldots, Z_t^K)\sigma \, d\tilde{B}_t \tag{5.7}$$

Given a contingent claim X in $L^1(\mathcal{F}_T, \mathbb{Q})$, we can use the martingale representation theorem [7, IV.36.5] to represent

$$V_t := \mathbb{E}_{\mathbb{Q}}(\beta_T X \mid \mathcal{F}_t) \quad \text{by} \quad \pi_0 + \int_0^t H_s . d\tilde{B}_s \quad (0 \le t \le T) \tag{5.8}$$

where $\pi_0 = \mathbb{E}_{\mathbb{Q}}(\beta_T X)$ and $\int_0^T |H_s|^2 \, ds < \infty$ a.s. We want to set $\phi := (\operatorname{diag}(Z)\sigma)^{-1\top} H$ and then have, by (5.8),

$$V_t = \pi_0 + \int_0^t \phi_s . dZ_s \tag{5.9}$$

but we have to pay attention to some technicalities.

As $\int_0^t |H_s|^2 \, ds$ and Z_t are continuous in t, we can assume, by localisation, that $\int_0^T |H_s|^2 \, ds$, Z_t and Z_t^{-1} are all bounded. We now need a

Technical Lemma. *If B is $\mathrm{BM}(\mathbb{R}^K)$ and A_t is a previsible matrix-valued process such that A_t and A_t^{-1} are bounded and $\phi_i \in L_2(B^i)$ where $L_2(X) = \{\text{previsible } G : \mathbb{E}(\int_0^T G_s^2 \, d[X]_s) < \infty\}$. Then $(A^\top\phi)_i \in L_2(\bar{a}_{ij} \cdot B^j)$, for all i, j, and where $A^{-1} = (\bar{a}_{ij})$, and also $A^\top\phi \cdot (A^{-1} \cdot B) = \phi \cdot B$.*

Proof

For ϕ bounded, we see that

$$(a_{ki}\phi_k) \cdot (\bar{a}_{ij} \cdot B^j) = (a_{ki}\bar{a}_{ij}\phi_k) \cdot B^j = \phi_j \cdot B^j = \phi \cdot B \tag{5.10}$$

Using summation convention, and [7, IV.27.6]. Otherwise

$$\int_0^T (A^\top\phi)_i^2 \, d[\bar{a}_{ij} \cdot B^j] = \int_0^T (a_{ki}\phi_k)^2 \bar{a}_{ij}^2 \, d[B^j] \le K \sum_j \int_0^T \phi_k^2 a_{ki}^2 \bar{a}_{ij}^2 \, dt$$

which has finite expectation as A and A^{-1} are bounded and $\phi_i \in L_2(B^i)$. So $(A^\top\phi)_i \in L_2(\bar{a}_{ij} \cdot B^j)$ (no sum), hence $(A^\top\phi)_i \in L_2((A^{-1} \cdot B)_i)$. So the swap used in (5.10) can be repeated using the density of bounded previsible processes in $L_2(B)$, and the L_2-continuity of $\phi \mapsto A^\top\phi$. $\square$

Setting A to be $(\operatorname{diag}(Z)\sigma)^{-1}$ which is bounded with bounded inverse, and as H is in $L_2(\tilde{B})$, the lemma tells us that ϕ is in $L_2(Z)$, and that (5.9) does hold.

Set $\phi_t^0 = V_t - \phi_t . Z_t$, then if we extend Z to $\bar{Z} = (1, Z)$ $(= \beta\bar{S})$, $\bar{\phi} = (\phi^0, \phi)$, then

$$V_t = \bar{\phi}_t . \bar{Z}_t = \pi_0 + \int_0^t \bar{\phi}_s . d\bar{Z}_s \tag{5.11}$$

So $\bar{\phi}$ is in some sense 'discount self-financing'. We want rather that it is really self-financing, that is

$$\bar{\phi}_t . \bar{S}_t = \bar{\phi}_0 . \bar{S}_0 + \int_0^t \bar{\phi}_s . d\bar{S}_s \tag{5.12}$$

Proof We use integration by parts, remembering that S^0 is of finite variation.

$$\begin{aligned} d(\bar{\phi}.\bar{S}) &= d(\bar{\phi}.(S^0\bar{Z})) = d(S^0(\bar{\phi}.\bar{Z})) = dS^0(\bar{\phi}.\bar{Z}) + S^0\, d(\bar{\phi}.\bar{Z}) + d[S^0, \bar{\phi}.\bar{Z}] \\ &= dS^0(\bar{\phi}.\bar{Z}) + S^0(\bar{\phi}.d\bar{Z}) + 0 \qquad \text{by (5.11)} \\ &= \bar{\phi}.(\bar{Z}\, dS^0 + S^0\, d\bar{Z} + d[S^0, \bar{Z}]) \\ &= \bar{\phi}.d(S^0\bar{Z}) = \bar{\phi}.d\bar{S} \end{aligned}$$

□

So ϕ manufactures claim X by trading in the stocks and the bond. Given initial investment V_0, no inflows are needed and no outflows are produced until time T. The value of the claim is thus

$$\pi_0 = \mathbb{E}_{\mathbb{Q}}(\beta_T X) \tag{5.13}$$

assuming there are no arbitrage opportunities.

For the case $X = (S_T^1 - c)^+$, or any other tractable example, we can perform the requisite integration, as we know $\mathbb{P}$, the Wiener measure, given, say, μ_0, σ_0 as in note (3) (Black-Scholes), and $d\mathbb{Q}/d\mathbb{P}$ is defined in (5.5, 5.6). A messy calculation does eventually show that $V_0 = C(S_0^1, 0)$ in the notation of section 4.

We see that the drift of the log price process is removed by the change of measure (along with the interest rate discount) showing mathematically why it is absent from the Black-Scholes formula. Economically one could argue that there will be a link between ρ, μ and σ to ensure long-term market equilibrium. For instance, if σ were small, we would expect μ to be close to ρ. So the formula depends on μ, but only through the interrelationships between the economic variables.

Note we can value all contingent claims X which are $\mathbb{Q}$-integrable, but we would be interested in a condition involving the more 'concrete' given measure $\mathbb{P}$. The following proposition supplies and a sufficient condition.

Proposition. (Rogers [6]) *If $X \in L^{1+\epsilon}(\mathcal{F}_T, \mathbb{P})$ then $X \in L^1(\mathcal{F}_T, \mathbb{Q})$, and is thus an attainable contingent claim.*

We use the fact that $\xi = d\mathbb{Q}/d\mathbb{P}$ is $L^q(\mathbb{P})$ integrable for all $q < \infty$, although it is not essentially bounded. Because

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ \int_0^T (-\sigma^{-1}\mu)\, dB_t - \frac{1}{2}\int_0^T |\sigma^{-1}\mu|^2\, dt \right\}$$

and $\sigma^{-1}\mu$ is bounded. Thus

$$\mathbb{E}_{\mathbb{Q}}(|X|) = \mathbb{E}_{\mathbb{P}}\left(|X|\,\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \le \|X\|_{1+\epsilon} \left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_{\frac{1+\epsilon}{\epsilon}} \qquad \text{Hölder}$$

Whence result. □

6 Continuous Case — Further Generalisations

We have seen that the continuous case, under the strong distributional assumption about the price processes, is quite easily managed. as we generalise the underlying distribution the situation becomes more complicated, and the assumptions become less economical and more on account of mathematical tractability.

Harrison and Pliska [3 §3] attempt a very general framework of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with semimartingale price process S. Drawing on their discrete case theorem that there can be no arbitrage if and only if there exists a measure under which the discounted price process is a martingale, they assume there exists a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$, under which βS is a $\mathbb{Q}$-martingale. ($\beta_t = 1/S_t^0$ as usual.) The problems, which they freely admit, arise when trying to define what is and what is not a trading strategy. Their definition is not economically intuitive and is sensitive to equivalent changes in the chosen measure $\mathbb{Q}$. (In the Brownian case of section 5, $\mathbb{Q}$ is unique, so these questions are less disconcerting.)

Nevertheless, Harrison and Pliska's approach is the most general reasonable attempt in the literature, so it is described here in essence.

H-P Assumptions

$(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \le t \le T\}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions. The price process $S = (S^0, S^1, \ldots, S^K)$ is an $\mathbb{R}^{K+1}$-valued, strictly positive adapted R-process (right continuous with left limits). Assume that the bond process S^0 is continuous and of finite variation. As usual $\beta_t = 1/S^0_t$ and $S^0_0 = 1$.

Also assume that there exists a measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which

$$Z := \beta(S^0, S^1, \ldots, S^K) \qquad \text{is a } \mathbb{R}^{K+1}\text{-martingale under } \mathbb{P}^*$$

This is simply a formal analogue of their discrete result that there are no arbitrage opportunities if and only if such a maringale measure exists. (Though we have seen that is true for the case in section 5.)

Then we define

$$\ell L^1(Z) := \left\{ \text{previsible } H = (H^1, \ldots, H^K) : \left(\int_0^\bullet (H^k_s)^2 \, d[Z^k]_s \right)^{\frac{1}{2}} \text{ is } \mathbb{P}^* \text{ locally integrable} \right\} \tag{6.1}$$

(Note on notation: Harrison and Pliska use the notation $\mathcal{L}(Z)$, but this notation is more in tune with the system of Jacod [5], which we shall be using later.) Their assumptions for a trading strategy $\phi = (\phi^0, \ldots, \phi^K)$ to be *admissible* are:

(1) ϕ^0 adapted and $(\phi^1, \ldots, \phi^K) \in \ell L^1(Z)$ (Does depend on choice of $\mathbb{P}^*$.)
(2) $V^*(\phi) \ge 0$, where $V^*(\phi) := \phi.Z$
(3) V^*_ϕ satisfies the SDE $dV^*_\phi = \phi.dZ$
(4) V^*_ϕ is a $\mathbb{P}^*$-martingale.

Notes on admissibility

(1) Allows the stochastic integral $\int \phi.dZ$ to be defined.
(2) An economic assumption.
(3) Makes sense as a self-financing condition for similar reasons as we saw in (5.12), except processes may not be continuous.

$$\Delta V_t(\phi) = S^0_t \Delta V^*_t(\phi) = S^0_t(\phi_t.\Delta Z_t) = \phi_t.\Delta(S^0_t Z_t) = \phi_t.\Delta S_t$$

as $\Delta V^*_t = \phi_t.\Delta Z_t$, so

$$V^*_{t-}(\phi) = V^*_t(\phi) - \Delta V^*_t(\phi) = \phi_t.Z_t - \phi_t.\Delta Z_t = \phi_t.Z_{t-}$$

The following completes the argument:

$$\begin{aligned} dV_t = d(S^0V^*)_t &= dS^0_t\, V^*_{t-} + S^0_t\, dV^*_t + d[S^0, V^*]_t \\ &= \phi_t.(dS^0_t\, Z_{t-}) + \phi_t.(S^0_t\, dZ_t) + \phi_t.d[S^0, Z]_t \\ &= \phi_t.(d(S^0Z)_t) = \phi_t.dS_t \end{aligned} \tag{6.2}$$

That is to say,

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_s.dS_s \tag{6.3}$$

That is that ϕ is self-financing in the discrete sense, that the changes in the value of the portfolio are due only to the changes in the prices of the securities, and not the outflow or inflow of funds. We can show the reverse implication, and hence deduce that condition (3) is independent of the measure $\mathbb{P}^*$

(4) $V^*(\phi)$ is a local martingale but condition (4) is also dependent on $\mathbb{P}^*$.

Then we say an element X of $L^1(\Omega, \mathcal{F}_T)^+$ is *attainable* if there exists an admissible ϕ such that

$$V_T(\phi) = X \qquad \text{and then} \quad V^*_T(\phi) = \beta_T X \tag{6.4}$$

And as $V^*(\phi)$ is a $\mathbb{P}^*$-martingale, then

$$V^*_t(\phi) = \mathbb{E}^*(V^*_T(\phi) \mid \mathcal{F}_t) = \mathbb{E}^*(\beta_T X \mid \mathcal{F}_t) \tag{6.5}$$

Thus

$$\pi = V^*_0(\phi) = \mathbb{E}^*(\beta_T X) = V_0(\phi) \tag{6.6}$$

as $\mathcal{F}_0$ is trivial. And in a viable market (no arbitrage), π is the only price at which security X should be traded at time 0, as X can be manufactured with trading strategy ϕ at time 0 at cost of $\pi = V_0(\phi)$. Note if there also another strategy ψ such that $V_T(\psi) = X$, then $V^*_t(\psi) = \mathbb{E}^*(\beta_T X \mid \mathcal{F}_t) = V^*_t(\phi)$, so the value of portfolio ψ equals that of ϕ at all times.

Given X such that $\beta_T X$ is $\mathbb{P}^*$-integrable, set

$$V^*_t = \mathbb{E}^*(\beta_T X \mid \mathcal{F}_t) \tag{6.7}$$

taking a modification to be an R-process. Then V^*_t is a positive ($\mathbb{P}^*$) martingale, and we see:

Proposition. *X is attainable if and only if V^* can be written*

$$V_t^* = V_0^* + \int_0^t H_s.dZ_s \tag{6.8}$$

for some $H \in \ell L^1(Z)$

Proof

'Only if' is obvious.

Conversely, suppose there is some such representation of V^*. Then set $\phi_t^0 = V_{t-}^* - \sum_{i=1}^K \phi_t^i Z_{t-}^i$, so that ϕ^0 is adapted (HP err here), and $V_t^*(\phi) = \phi_t.Z_t = V_{t-}^* + \sum_{i=1}^K \phi_t^i \Delta Z_t^i = V_t^*$ by (6.8). Thus ϕ satisfies the admissibility conditions (1)–(4) with

$$V_T^* = \mathbb{E}(\beta_T X \mid \mathcal{F}_T) = \beta_T X, \qquad \text{so} \quad V_T(\phi) = X$$

□

So *completeness* (attainability of all integrable $\beta_T X$) of the market depends on the representation (6.8) of the martingale V^*. In section 5, we used the martingale representation theorem and properties of the price process to achieve this.

Harrison and Pliska in their follow-up note [4] show:

Theorem. *The following are equivalent:*
(1) $\mathbb{P}^*$ is unique.
(2) every $\mathbb{P}^*$ martingale M can be written $M = M_0 + H \cdot Z$ for some H in $\ell L^1(Z)$.
(3) the market model is complete under $\mathbb{P}^*$.

Proof

(2) implies (3) by (6.8)

(3) implies (2) given (6.8) and the observation that any martingale can be written as the difference of positive martingales.

To show that (2) is equivalent to (1). In the language of Jacod [5], we set

$$M(Z) = \{ \text{probability measures } P \text{ on } (\Omega, \mathcal{F}) \text{ such that } Z \text{ is a } P\text{-martingale} \}$$
$$H^1(P) = \{ M : M \text{ is a } P \text{ martingale such that } M_T^* \text{ is } P\text{-integrable} \}$$
$$L^1(Z) = \{ H : H \text{ previsible and } \left(\int_0^T {H_s^k}^2 \, d[Z^k]_s \right)^{\frac{1}{2}} \text{ integrable} \}$$

(Here $M_t^* := \sup\{ |M_s| : 0 \le s \le t \}$.) Then Jacod shows [5, 11.2–11.4] that the following are equivalent for any $P \in M(Z)$:
(a) $P \in M_e(Z)$, the set of extreme points of $M(Z)$.
(b) $H^1(P) = \{ H \cdot Z : H \in L^1(Z) \}$.
(c) If $Q \in M(Z)$ and $Q \ll P$ then $Q = P$.

Condition (b) asserts that all 'well-behaved' M are representable by equally 'well-behaved' previsible processes. Localisation allows us to make (b) equivalent to condition (2) of the theorem.

Lemma A. *Condition (b) holds for $\mathbb{P}^*$ if and only if condition (2) holds.*

Proof of lemma A

Suppose (2) holds. Remember the generalised Davis, Burkhölder, Gundy inequality [5, 2.34], that for $1 \le p < \infty$ there exist constants c_p, c_p', such that for all local martingales M,

$$||M_\infty^*||_p \le c_p || \, [M]_\infty^{\frac{1}{2}} ||_p \le c_p' ||M_\infty^*||_p \tag{6.9}$$

Then if $M \in H^1(\mathbb{P}^*)$, (2) gives $M = H \cdot Z$ for H previsible and $[H \cdot Z]^{\frac{1}{2}}$ locally integrable. But

$$|| \, [H \cdot Z]_T^{\frac{1}{2}} ||_1 \le \frac{c_1'}{c_1} ||M_T^*||_1 < \infty$$

So H is in $L^1(Z)$ and $M \in \{ H \cdot Z : H \in L^1(Z) \}$.
If $M \in \{ H \cdot Z : H \in L^1(Z) \}$ then M is a local martingale with M_T^* integrable by (6.9), so M is a martingale, hence (b) holds.

Conversely, if M is a $\mathbb{P}^*$ martingale, define a sequence of stopping times T_n by $T_n = \inf\{ t : M_t^* > n \}$. Then $(M^{T_n})_T^* \leq |M_T| + n$ which is integrable. So M^{T_n} is in $L^1(\mathbb{P}^*)$, hence $M^{T_n} = H^n \cdot Z$ for some H in $L^1(Z)$. Pasting together $M = H \cdot Z$, and H is locally $L^1(Z)$, that is, is in $\ell L^1(Z)$, because $H_t^n = H_t$ for $t \leq T_n(\omega)$. □

Now back to showing (2) $\iff$ (1). Suppose (2) holds for $\mathbb{P}^*$, then (b) and hence (c) hold for $\mathbb{P}^*$. Hence if $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}^*$, under which Z is a martingale, then $\mathbb{Q} \in M(Z)$ and $\mathbb{Q} \ll \mathbb{P}^*$ so $\mathbb{Q} = \mathbb{P}^*$. Hence $\mathbb{P}^*$ is unique, and so (1) holds.
Conversely, suppose (1) holds and (2) does not. Then neither (a) nor (b) holds either for $\mathbb{P}^*$. So $\mathbb{P}^*$ is not extreme in $M(Z)$. Thus

$$\mathbb{P}^* = \alpha Q_1 + (1-\alpha) Q_2 \qquad Q_1, Q_2 \in M(Z) \quad 0 < \alpha < 1 \tag{6.10}$$

Hence $Q_1 \leq \frac{1}{\alpha}\mathbb{P}^*$. We need the following result:

Lemma B. *If Z is a non-negative P martingale and Q a measure such that $Q \leq kP$ for some constant k, and Z is a Q-local martingale, then Z is a Q-martingale.*

Proof of lemma B

Fatou's lemma shows that Z is a Q-supermartingale, so we only have to show that $\mathbb{E}_Q(Z_t) = Z_0$ for all t to get the result. Fix t, and let $\mathcal{C} = \{ Z_T : T \text{ stopping time } \leq t \}$. We know that $\mathcal{C}$ is P-uniformly integrable as Z is a P-martingale. But

$$\sup_{X \in \mathcal{C}} \int_{|X|>a} |X| \, dQ \leq k \sup_{X \in \mathcal{C}} \int_{|X|>a} |X| \, dP \to 0 \qquad \text{as } a \to \infty$$

So $\mathcal{C}$ is Q-uniformly integrable. Now take the sequence of stopping times T_n which reduce Z for Q, that is $T_n \uparrow \infty$ and Z^{T_n} is a Q-martingale. Then $Z_t^{T_n} \to Z_t$ and $\{ Z_t^{T_n} = Z_{t \wedge T_n} \}$ is Q-uniformly integrable. Thus $Z_0 = \mathbb{E}_Q(Z_t^{T_n}) \to \mathbb{E}_Q(Z_t)$ □

So by Lemma B, Z is a Q_1-martingale, and also a Q_2-martingale. So Z is a $(\beta Q_1 + (1-\beta)Q_2)$-martingale for $0 < \beta < 1$. But $(\beta Q_1 + (1-\beta)Q_2)$ is equivalent to $\mathbb{P}^*$, contradicting the uniqueness of $\mathbb{P}^*$. □

So a market model could be said to be good (both viable and complete) if there exists only one 'martingale measure' $\mathbb{P}^*$, then all contingent claims X are valued by

$$\pi = \mathbb{E}^*(\beta_T X) \tag{6.11}$$

7 Other Futures Contracts

The European call option which we have analysed before is only one in a wide class of *futures contracts*, agreements whose outcome depends on a (stochastic) future. The agreement in this case is that the buyer of the option has the right, but not the obligation, to buy a share in the stock S^1, at a given price c, at a given future date T. Thus the buyer has a control theory problem of a very simple type. At expiration, his net gain due to exercising the option or not is:

$$gain = \begin{cases} S_T^1 - c & \text{if option exercised} \\ 0 & \text{if option lapses} \end{cases} \tag{7.1}$$

Clearly the optimal strategy is to exercise the option if and only if $S_T^1 > c$, then the gain is

$$(S_T^1 - c)^+, \tag{7.2}$$

which is the familiar random variable associated with the option, and which we used in our valuation formulae. The reason for writing out the obvious in a longwinded fashion is that for some contracts the optimisation of strategy is non-trivial.

For instance, the *American call option* gives the right to buy the stock at a given price at any time between now and the expiration date T. Dynamic stochastic programming indicates that the best strategy is to exercise the option at time t if $S^1_t > c_t$ where $(c_t : 0 \le t \le T)$ is some deterministic process. In the Black-Scholes model, we had $C(S^1_t, t)$ as the price of a European call option at time t. We saw that $\frac{\partial C}{\partial t} < 0$ and thus deduce, as Black-Scholes themselves remark [1], that

$$C(S^1_t, t) > C(S^1_t, T) = (S^1_t - c)^+ \ge S^1_t - c \qquad (t < T) \tag{7.3}$$

Which means, the European option is worth more than the gain due to buying the stock for price c now. We deduce that as an American call must be at least as good as a European call, the value of an American call at time $t < T$ must be strictly greater than the gain realised through exercising the option at that moment. So the optimal strategy is to wait until expiration and exercise the option then if and only if $S^1_T > c$ as before. That is, the best thing to do with an American call is to pretend that it is a European call, so they share the common value $C(S^1_0, 0)$ at time 0.

In arbitrage language, suppose we sell Mr Silly an American call on stock S^1 at time 0 for $C(S^1_0, 0)$, which we use the buy the portfolio which will produce the European call $(S^1_T - c)^+$ at time T. Then, were Mr Silly to exercise his option at some time $t < T$, we would have to give him a share of the stock at a cost of $S^1_t - c$ to us. However the value of the portfolio at that time would be $C(S^1_t, t)$ which, by (7.3) is greater than our cost. The point here is that although we did not plan to manufacture $S^1_t - c$, we can still produce it and even have a little profit left over as Mr Silly's decision was not optimal.

Another standard contract is the *European put option*, under which the buyer of the option has the right (but not the obligation) to sell a share to the option-seller at time T for a fixed price c. By a similar argument to the European call (more basic control theory), its value at time T is seen to be

$$(S^1_T - c)^- \tag{7.4}$$

We value the option before time T with the following trick, due to Black-Scholes. Buy a call option and sell a put option, so the portfolio's value at time T is

$$(S^1_T - c)^+ - (S^1_T - c)^- = S^1_T - c \tag{7.5}$$

This is easily manufactured by buying a share at time 0 and borrowing ce^{-rT}, so that at time T the share is worth S^1_T and the debt has incerased to c. This requires an initial investment of $S^1_0 - ce^{-rT}$. So if $D(S^1_0, 0)$ is the value of the put option at time 0,

$$C(S^1_0, 0) - D(S^1_0, 0) = S^1_0 - ce^{-rT} \tag{7.6}$$

Hence

$$\begin{aligned} D(x,t) &= C(x,t) - x + ce^{-r(T-t)} \\ D(x,t) &= -x\Phi\left(-\frac{\log\frac{x}{c} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) + ce^{-r(T-t)}\Phi\left(-\frac{\log\frac{x}{c} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{7.7}$$

$D(x,t) \to (x-c)^-$ as $t \to T$, but not monotonically like $C(x,t) \downarrow (x-c)^+$. Crucially $D(\cdot,t) \not> D(\cdot,T)$, as is the case for C. This means that our previous argument does not apply to the *American put option*, which gives the right to sell at any time up to T. Thus the American put presents a difficult optimal control problem. Suppose an American put should be the same price as a European put. Then buy one and follow the strategy: behave as if holding a European put unless there is a time $t < T$ such that $D(S^1_t, t) < c - S^1_t$ (which happens with positive probability). Then overall the amount we get from the option-seller's portfolio is at least as large as the European put cash stream, but is strictly greater with positive probabilty. This is not viable for a long-term market. Thus we deduce that an American put is worth strictly more than a European put.

The common contract, often referred to simply as a *future*, is an agreement between two agents for one to buy from the other a unit of some stock or commodity at some given future date for a price decided now. The future contract has no consideration on either side as this can be reflected in the contract price. in the context of our model this contract is trivial,as the buyer's gain at time T is simply

$$S_T^1 - c$$

where S^1 is the commodity price process and c is the contract price. The buyer manufactures the desired amount by buying one share at cost S_0^1 and borrowing ce^{-rT} as before. Thus the future price c should be fixed by

$$c = e^{rT} S_0^1 \tag{7.8}$$

That is, c is the current price of the commodity inflated by the known riskless interest rate. This would imply that no futures trading should ever be done, as it is equivalent to simply buying the share now, but for less money.

However, if the commodity is, say, spring potatoes and it is currently winter, the commodity is not available and can only be traded in the 'future'. Or if the commodity were fresh eggs and T were one year in the future, one could not buy the eggs now which would honour the contract in a year's time. Even for financial securities (no holding costs or depreciation and always marketed), real-world transaction costs translate our equation (7.8) into inequalities, merely constraining the range of the contract price. Futures allow markets to gamble on future events which may not be already fully discounted in the stock price, and also to deal in composite commodities, such as a FTSE future comprising 100 major UK equities.

If the commodity has a deterministic storage cost, c_t, at time t, then (assuming we can have a negative holding of the commodity incurring negative storage costs) its future price must be:

$$c^* = e^{rT} S_0^1 + \int_0^T c_t e^{r(T-t)}\, dt \tag{7.9}$$

by construction of cash streams to pay storage costs. This is found in Duffie [2, Ex22.17].

An important application, described by Black-Scholes [1], is that of valuing corporate liabilities. If a company issues bonds at a face value of c to be redeemed at a fixed future time T, the shareholders have the choice at time T either to honour the bonds and pay out c, or to let the company be wound up. This is equivalent to the shareholders having an option on the value of the company with exercise price c. Then if the company is worth S_t^1 in total at time t, the shares are worth $C(S_t^1, t)$ (which is greater than $(S_t^1 - c)^+$, and the bonds are worth $S_t^1 - C(S_t^1, t)$ which is less than c.

8 Conclusion

In the continuous case under the no arbitrage assumption we have considered several models. In the simplest price diffusion process [Section 4] all contingent claims were attainable in the sense of being able to be manufactured from basic securities so that their price is unique and calculable. A more general higher-dimensional model [Section 5] provides a similar result. The most general approach of Section 6 shows that attainability is equivalent to a martingale representation problem which is generally solvable if and only if the martingale measure for the discounted price process is unique. If this is not so, we have seen that it even becomes difficult to define trading strategies, as the definitions depend on the measure chosen.

We have assumed no dividend streams or cash calls relating to the stocks, but these could be included in the general case by adding or subtracting the appropriate discontinuous jumps to the price processes. The absence of transaction costs and taxation is less easily dealt with. The unboundedness of the trading strategies has not been too bothersome as in both European call and put options the amount of stock held or sold short is always between 0 and 1. Cash holdings or borrowings are less than the discounted stock price which is L^2-bounded in the simpler models.

The admissibility conditions for trading strategies unarguably owe more to mathematical expediency than economic realism. The boundedness condition (ℓL^1) is very weak, perhaps too weak for reality, but the martingale assumption is on the strong side. A relaxed attitude is to view the complete perfect market of our model as an approximation to the real world of complex stochastic interaction, transaction costs, and so forth. This attitude is boosted by the result in [1] showing that the results of the model do come close to actual security prices in the market.

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