# Unfair Games


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"Fair is foul, and foul is fair" cry the witches in *Macbeth* warning against reliance on surface appearances, a recurring theme throughout the play. Taking the theme, here presented are some games, which themseleves are not as fair as they might seem. The skill of the games lies not with its playing but with the analysis and understanding of it, for we shall see that the outcome (even of random games) was never in doubt – but foully predetermined.

**BRUSSELS SPROUTS: A topological unfair game**

Many readers will be familiar with the two player game of *Sprouts*, in which every move extends a graph. (A graph is a set of points (vertices) joined together by non-intersecting lines (edges) on some topological surface, such as a sheet of paper.) Starting with two unconnected vertices, a move consists of drawing a new edge from one existing vertex to another, or even back to itself. The only restriction is that no vertex can have more than three edges connected to it (counting double for looping back edges). The new edge cannot cross itself or an existing edge. The move is completed by creating a new vertex half-way along the new edge, splitting it into two edges, and leaving the new vertex with the capacity for only one more edge to be connected to it. Players move alternately, and the winner is the last player able to make a valid move. Figures 1 and 2 are two sample games:

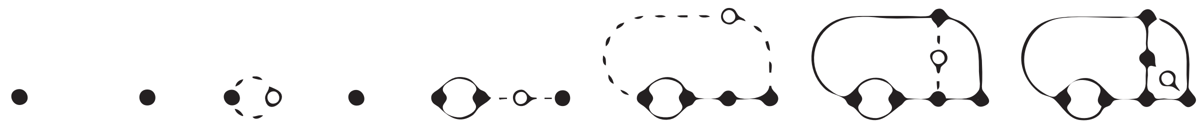


**Figure 1.** Player 1 wins

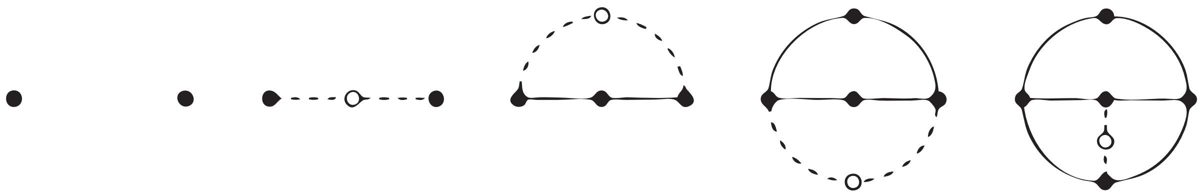


**Figure 2.** Player 2 wins

By noting that the total capacity of the graph to accept new edges decreases by one with each move, we see that the game must finish in five moves or less. The game can be made longer and more complex by increasing the number of initial vertices. In fact the 2-sprout game is (fairly) easily analysed and an intelligent player 2 will always win.

In *Brussels Sprouts* we replace the dot shaped vertices with cross-shaped ones (+), and insist that all edges be attached to a free point of some cross. The new vertex placed on a new edge should have one bar of the cross lying along the line, and a cross-point lying on either side. Figure 3 demonstrates a game with one initial vertex:

**Figure 3.** Player 1 wins

It is not immediately apparent that this does not just complicate the game further (it certainly lasts longer), but in fact the analysis is greatly simplified. This is because the game is determined by the faces of the graph. A face is a connected region of the space closed off from the other faces by the edges of the graph. We also consider the region of space outside the graph as a face. Each face must have at least one free cross-point in it, as the newest boundary edge of the face left a free cross-point on each side. If there is a face with more than one free internal cross-point, then the game is not yet over as one valid move is just to join them and split the face into two. So the game ends when there is exactly one free cross-point in each face. Each move uses up two cross-points, but also creates two new ones, leaving the total unchanged. Thus in the 2-sprout game, the game ends whenever there are 8 faces of the graph. To discover when this is, we use *Euler's Formula*:

$$V - E + F = 2, \tag{1}$$

where $V$ = number of vertices, $E$ = number of edges, $F$ = number of faces. (The precise conditions for validity will be discussed later.) After $n$ moves these values will be:

$$\begin{aligned} V_n &= n + 2 \quad \text{(each move adds a new vertex)} \\ E_n &= 2n \quad \text{(each move creates a new edge which splits into two)} \\ \text{thus} \quad F_n &= n \quad \text{(by Euler's formula)} \end{aligned} \tag{2}$$

We know that the game ends when and only when $F_n = 8$. Thus (by Equation 2) the game always ends after exactly 8 moves, and player 2 will always win, no matter what either player does during the game. Try this for yourself now for a couple of games; you will always finish with 10 vertices and 8 faces.

Now if we start with $m$ vertices, then:

$$V_n = n + m \quad E_n = 2n \quad F_n = 2 - m + n \tag{3}$$

The game ends when $F_n = 4m$, that is when $n = 5m - 2$. So player 1 wins when $m$ is odd, and player 2 wins when $m$ is even. It is easy to confuse the unwary opponent by changing the parity of $m$ if he or she becomes suspicious of the order of play. The situation can be further complicated by having $k$ players instead of two, the winner being the last to make a valid move. Then player $n$ will win if and only if:

$$n \equiv 5m - 2 \pmod{k} \tag{4}$$

Luckily 5 is prime, so for any $k$ not divisible by 5, there is a range of values of $m$ to make any given player the winner.

The *coup de grâce* is dealt by playing the game on more interesting toplogical spaces than flat paper, such as the torus, Möbius strip, sphere, and so on. These surfaces are characterised topologically by their number of "handles" (*genus*), and their number of "crosscaps". The generalised Euler formula is:

$$V - E + F = 2 - 2g - h, \tag{5}$$

where $g$ is the number of handles, and $h$ the number of crosscaps (Möbius strips sewn into the surface). The following remarks on the validity of this formula may safely be ignored.

* * * * *

Strictly Euler only applies to a graph embedded in a closed combinatorial surface. So the graph must be connected, and all its faces simply connected.

(1) The final graph will be connected, as any two components would have a free cross-point in the outer region, which could be joined as a next move.

(2) The knowing player must ensure by the end of the game that all the faces are simply connected, as there is no guarantee that this need happen otherwise. In practice this means using all the wrapovers and points of commonality discussed later. For example, on the torus there should be edges going round the circles shown in figure 4.

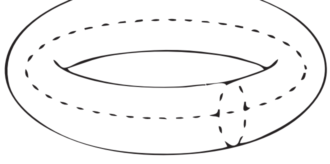


**Figure 4.** Torus

(3) It is certainly true for connected closed combinatorial surfaces, which are exactly the connected compact 2-manifolds. And these, apart from the sphere, are either a sphere with $g$ handles, or a sphere with $h$ crosscaps. (A sphere with $g$ handles and $h$ crosscaps is equivalent to one with $2g + h$ crosscaps, $g, h \neq 0$.)

Euler is also valid for $\mathbb{C}$ (the complex plane), its open connected subsets, cylinders, and many other two dimensional surfaces, as they can all be mapped onto the 2-sphere for which Euler is valid. Table 1 describes some ”common” surfaces.

* * * * *

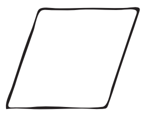
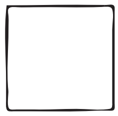
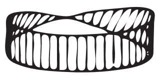
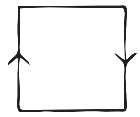
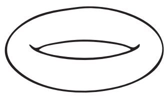
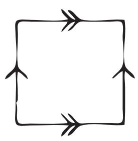
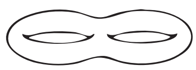
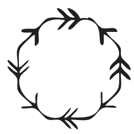
| Surface | Picture | Representation | | $g, h$ |
|---|---|---|---|---|
| $\mathbb{C}$-plane, sphere, etc. |  |  | sheet of paper | $0, 0$ |
| Möbius strip |  |  | square with one pair of opposite sides reversely identified | $0, 1$ |
| Torus |  |  | square with opposite sides identified | $1, 0$ |
| Double torus |  |  | octogon formed by joining two tori | $2, 0$ |

**Table 1.** Surface Summary

So at the end of the game:

$$n \equiv 5m + h + 2g - 2 \pmod{k} \tag{6}$$

With three degrees of freedom $(m, g, h)$, a wise huckster could keep opposition confusion going for an impressive amount of time playing with twisted sprouts.

**NIM - unfair in binary**

*Nim* has been understood since Charles Bouton's paper of 1901, so there is nothing new in this account, which is included for completeness and as an illustration.

In *Nim* there are a number of rows, each with a number of counters along it. A traditional starting pattern is to have $n$ counters in the $n^{th}$ row (figure 5), but this does not matter much.

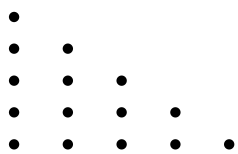


**Figure 5.** *Nim* start position

A move is simply to choose a row, and to remove from it as many counters as desired. The winner is the player who removes the last counter. There is a suprisingly simple function of the position which reveals which player can win.

Many will know the logical binary operators AND ($\wedge$) and OR ($\vee$), perhaps fewer the operation EOR ($\veebar$) (or XOR) - the exclusive-OR. Its truth table is shown in table 2.

| **Table 2.** Exclusive-OR | | |
|---|---|---|
| $a$ | $b$ | $a \veebar b$ |
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

We can extend EOR (and the other operations) from just acting on $\{0,1\}$ to acting on all non-negative integers, via their binary expansions. That is:

$$\text{If } a = \sum_{n=0}^{\infty} a_n 2^n \quad b = \sum_{n=0}^{\infty} b_n 2^n, \quad \text{then } a \veebar b = \sum_{n=0}^{\infty} (a_n \veebar b_n) 2^n \tag{7}$$

(In fact $(\{0,1\}, \veebar, \wedge)$ is the field $\mathbb{F}_2$, and we create the commutative product ring $(\{0,1\}^{\mathbb{N}}, \veebar, \wedge)$ under componentwise operations, the non-negative integers being a subring. So $(\{0,1\}^{\mathbb{N}}, \veebar)$ is an abelian group of elements of order 2.)

Returning to *Nim*, if we represent the state of the game as $r_1, r_2, \ldots$ where $r_n$ is the number of counters left in the $n^{th}$ row, then we consider $R = r_1 \veebar r_2 \veebar \ldots$ What does it mean when $R = 0$ ?

(i) There are at least two non-empty rows, so the player about to play is not just about to win.

(ii) Once that player has moved, $R$ will no longer be zero. For if he reduces row $n$ from $r_n$ to $r'_n$ counters, then $R$ changes to $R'$:
$$R' = R \veebar (r_n \veebar r'_n) = r_n \veebar r'_n \neq 0 \tag{8}$$

(iii) We shall prove shortly that whatever the move was, there is a move by the other player which returns $R$ to zero.

So if player 1, say, can arrange for $R$ to be 0 after his move, then player 2 cannot win with his next move and player 1 can return $R$ to 0. Thus player 2 can never win, so player 1 must win (as the game always ends).

Let us quickly see how to return $R$ to 0:

Let $k$ be the position of the leading '1' in the binary expansion of $R'$ (non-zero). Then there exists an $n$ such that $r_n$ has its $k^{th}$ bit set, as $R'$ indicates that there is an odd number of such $n$. Set $r'_n$ to be $R' \veebar r_n$, then the bits of $r'_n$ higher than the $k^{th}$ are unchanged as $R'$ is zero there, and the $k^{th}$ bit itself is zero. Thus $r'_n < r_n$, so the move to make is to remove $r_n - r'_n$ counters from the $n^{th}$ row, as $R'' = R' \veebar (r_n \veebar r'_n) = R' \veebar R' = 0$.

The above shows how to make $R$ equal to zero given any position in which it is not. So a knowledgeable player 1, if the initial value of $R$ is non-zero, will win whatever player 2 does. Similarly a knowledgeable player 2 will win if $R$ is initially zero. If one of the players is unknowledgeable he or she will probably leave $R$ in a non-zero state after some move, enabling the other to zero $R$ and take the advantage.

We see that the game is determined by the initial value of $R$, $R_0$. For a starting position of $n$ rows with $i$ counters in the $i^{th}$ row, $R_0 = \veebar_{i=1}^{n} i$. Now $(2m) \veebar (2m+1) = 1$ so we deduce:

$$R_0 = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4 \\ n+1 & \text{if } n \equiv 2 \bmod 4 \\ 0 & \text{if } n \equiv 3 \bmod 4 \\ n & \text{if } n \equiv 4 \bmod 4 \end{cases} \tag{9}$$

Player 2 can win when 4 divides $(n-3)$, player 1 otherwise.

In a sense both *Nim* and *Brussels Sprouts* are no more unfair than any other discrete (turn-about) two-player games, such as chess. For, as long as the game always ends, there will be a strategy for one of the players which gives victory no matter what the other does. In *Brussels Sprouts* we saw that the winner is determined by the initial number of vertices and the winning strategy is arbitrary, as it does not matter at all. In *Nim* the position is characterised by the value of $R$, and the simple winning strategy is described above. In chess, say, there are 3 outcomes (win, lose, or draw) yet an optimal strategy for either black or white does exist. (History suggests that it could be white which is always able to make at least a draw, but it is an open question.) However it is doubtful whether it is governed by some simple rule or formula and the computation involved in calculating it (and it is computable in a technical sense) would be substantial - there are of the order of $10^{50}$ positions.

**DOUBLE OR QUITS - walks around random examples**

Now we consider the result of a random game to be the (real-valued) net gain $X$ of total rewards or winnings less any stake, penalty or cost due to playing the game. If the game is voluntary, with gain $Y$ due to not playing the game, we shall consider $X - Y$, the relative gain of playing the game. Knowing the distribution of the net gain $X$, we want to know if the game is worth playing, or at least fair. Two simple alternative questions to ask are:

$$\text{Is} \quad \mathbb{P}(X \geq 0) \geq \frac{1}{2}\ ? \quad \text{Is} \quad \mathbb{E}(X) \geq 0\ ? \tag{10}$$

(We are assuming that $X$ is an integrable random variable on some probability space.) These distinct conditions are often confused. For example, even Adam Smith asserts of a lottery that:

> *there is not, however, a more certain proposition in mathematics than that the more tickets you adventure upon, the more likely you are to be a loser.*
>
> Adam Smith, *Wealth of Nations, I.x.i*

Consider a lottery of 1000 tickets each at unit cost, with only one prize of 900. Buying $n$ tickets, with a resultant net gain of $X_n$, we see that:

$$\mathbb{P}(X_n \geq 0) = \begin{cases} \frac{n}{1000} & \text{if } 1 \leq n \leq 900 \\ 0 & \text{if } 900 < n \leq 1000 \end{cases} \tag{11}$$

$$\mathbb{E}(X_n) = -\frac{n}{10} \tag{12}$$

So the probability of being a net winner increases with the number of tickets bought (up to a limit), but the expected gain decreases. As no-one ever made a living by winning lotteries, one might assume that the expectation is the indicator that the wise should use. One could further ask why so many people bet on horse races, play slot machines, and enter the football pools; all activities with negative expectations of gain. Prof Blimp would inform you that these people have no head for such things and have conquered rationality by a combination of greed and the natural human belief in one's own good fortune. His sociology colleague Dr Larebil suggests that what is being sold is not just the chance of a fortune but the opportunity to dream and to hope about winning it. One former Chancellor of the Exchequer clearly agrees:

> **Lawson:** *Well, the sum you might win is absolutely enormous*
> **Question:** *But the odds are enormous too.*
> **Lawson:** *Yes, I know, but it's worth doing just for the money, just for the possibility of getting this enormous sum.*
> William Keegan, *Mr Lawson's Gamble*, Hodder and Stoughton 1989

This dream price can be the whole cost, as in Big Brother's lottery for the proles in Orwell's *1984*, which never really awarded the large prizes that it claimed to. Big Sister's premium bonds at least satisfy $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}(X) > 0$ for a unit stake over one year. In fact $\mathbb{E}(X) = 0.07$ (7% interest rate), which implies a negative relative gain, when compared with investing in a bank or building society with $\mathbb{E}(X)$ nearer 0.09 (or 9% net interest).

However Prof Blimp takes care each year to renew the insurance of his house against fire, a game where he is certain to lose his stake (premium) each year for no net gain, that is $X = -1$ *a.s.* However, if he did not play the game, that is if he failed to insure his house, his gain $Y$, would be 0 with a high probability, but would be minus the value of his house with the small (but positive) probability that he leaves the gas on, or some other conflagrationary accident occurs. So his relative gain $X - Y$ is probably slightly negative, but has a chance of being very large and positive. As the insurance company has to make provision for overheads and profits, the premium will be more than the value of the house multiplied by the probability of its incineration: $\mathbb{E}(X - Y) < 0$. We now see that the relative gain of playing the football pools has exactly the same stochastic structure as the relative gain of insuring a house.

I am not however advocating either the purchase of 1000 pools coupons each week, nor the cancellation of insurance policies, and neither will *Eureka* accept any liability for any losses incurred as a result of such action—for there is a difference between the two cases. In the former, you pay to have a tall but probability-thin spike added to your ‘gain-space’, in the latter you pay to have a similar, but negative, spike removed. In one the maximum gain is made much greater, in the other the minimum gain is made much greater (the maximum loss is made much less). The examples are here to show that, in real situations, even the stochastic structure of the gain random variable may not be enough to decide the question.

To conclude, I present a general case where the probability of success and the expectation of reward differ in a staggeringly paradoxical fashion.

Consider a sequence of random win/lose games agaainst a ‘house’ or bank. The player only chooses the stake which linearly scales the gains and losses of the game. The stake can be zero (but not negative), allowing the player to stop playing at any time, by setting the stake to be zero from then on. The stake can depend on the results of previous games, but cannot depend on the results of games yet to come—the bank does not allow betting after the event. The games are probabilistically independent, with the $n^{\text{th}}$ game yielding probability $p_n$ of success with a reward $\alpha_n > 1$ for a unit stake. So the gain $X_n$, of playing game $n$ is:

$$
\begin{aligned}
X_n &= \begin{cases} \alpha_n - 1 & \text{with probability } p_n \ (0 < p_n < 1) \\ -1 & \text{with probability } q_n = 1 - p_n \end{cases} \\
\text{and} \quad M_n &= \sum_{r=1}^{n} C_r X_r
\end{aligned}
\tag{13}
$$

is the total gain so far, where $C_n$ ($\geq 0$) is the stake placed on game $n$. The *Borel-Cantelli Lemmas* imply that the probability of the player having infinitely many victories is either 0 or 1, depending on whether the sum $\sum p_n$ converges or diverges. Unbelievable paradoxes arise in either case.

*Case 1.* $\sum p_n < \infty$, *certain to have only finitely many victories.*

Suppose $\alpha_n = \frac{2}{p_n}$, then $\mathbb{E}(X_n) = 1$, and we choose to gamble a unit stake each turn, $C_n = 1$, then we see:

$$M_n \to -\infty \text{ with probability } 1 \qquad \text{but} \quad \mathbb{E}(M_n) \to +\infty$$

*Case 2.* $\sum p_n = \infty$, *certain to have infinitely many victories.*

The gambler can choose each turn to stake exactly the amount needed to restore all past losses and give a small profit if victorious in the game:

$$C_n = \frac{n - M_{n-1}}{\alpha_n - 1} \qquad \text{so } M_n = n \text{ if successful.}$$

This results in a fortune strictly increasing from one victory to the next, and if the gambler chooses to stop after the $k^{\text{th}}$ victory (and there will be a $k^{\text{th}}$ victory with probability 1), then he finishes with a positive net gain. A slightly simpler case is $C_n = (N - M_{n-1})/(\alpha_n - 1)$ which stops after the first victory with a profit of $N$, for arbitrary ‘greed value’ $N$. This is a generalisation of the stake doubling rule commonly associated with roulette, under which the initial bet is 1, which is doubled after every losing spin of the wheel, so that the eventual victory yields a net gain of 1. In this case $p_n = q_n = \frac{1}{2}$, $\alpha_n = 2$, and $C_n = 2^{n-1}$ until victory. By picking

$$\alpha_n = \frac{p_n + 1}{2p_n} \quad \text{we get} \quad \mathbb{E}(X_n) = -\tfrac{1}{2} q_n, \ \text{so } \mathbb{E}(M_n) < 0.$$

So we have a negative expectation of $M_n$, yet $M_n$ itself tends to some positive value $N$ with probability one.

In both these cases, the limiting value of $M_n$ gives the true worth of playing the game, and the expectations are misleading. The games are not however as good for the theoretical winner as might be supposed. In the former, where the bank wins, the amount “invested” in paying out to the player on his finite number of wins has infinite expectation, so the bank needs deep pockets. Alternatively if only a score is being kept, the real winner is the player who does not keep score, as an infinite amount of paper will have to be bought to record it.

In the latter game, in the simple stop-after-first-win case, the expected losses before the first win can have infinite expectation. (If $\alpha_n = 1/p_n$, then $\mathbb{E}(M_n) = 0$ and $\mathbb{E}(|M_n|) \leq N$.) The only snag now is that although $\mathbb{P}(T\infty) = 1$, where $T$ is the time of the first victory, it may be that $\mathbb{E}(T) = \infty$, that is you should expect an infinitely long wait before success. For example,

$$p_n = \frac{1}{n+1}, \quad \mathbb{P}(T \geq n) = \frac{1}{n}, \quad \mathbb{E}(T) = \sum_{n \geq 1} \mathbb{P}(T \geq n) = \infty.$$

## REFERENCES AND ACKNOWLEDGEMENTS

**Sprouts:** I am indebted to Eric Brown of Moray House College of Education in Edinburgh who introduced me to topology via Brussels Sprouts, and to Andrew Wren of Cambridge for checking my topological statements.

**Nim:** More on Nim can be found in Chapter 15 of

Martin Gardner, *Mathematical Puzzles and Diversions*, Pelican 1965.

**Random Games:** is composed of snippets stolen from lectures, example sheets, and other sources.